© $U = \{2xe_1 + xe_2 + 3xe_4 \in V \mid x \in \mathbb{R}\}$

$W \oplus U$

- $(2,1,0,3)$ comp. di U risp. B_1

  $\dim U = 1$

- 3 vett.: $((0,0,1,0),(1,0,0,0),(0,1,0,0)) = B_W$
- $((2,1,0,3),(0,0,1,0),(1,0,0,0),(0,1,0,0))$

  $B_U + B_W$ … è libero

  $U + W = V$

$W = \{(\alpha,\beta,\gamma,0) \Rightarrow \alpha e_1 + \beta e_2 + \gamma e_3 \in V \mid \alpha,\beta,\gamma \in \mathbb{R}\}$

- $W + U = V$
- $\dim W = 3$ (t. di Grassmann)

  $\dim W \cap V = 0 \Rightarrow W \cap U = \underline{0}$

  $\Rightarrow W \oplus U.$

ES.1: $M_2(\mathbb{R})$

$$A = \left\{ \begin{pmatrix} \alpha & -2 \\ -2\alpha & 0 \end{pmatrix} \middle| \alpha \in \mathbb{R} \right\}$$

(a) det. $L(A)$, una base e dimens.;

(b) det. le componenti della matrice $\begin{pmatrix} 2 & 5 \\ -4 & 0 \end{pmatrix}$ di $L(A)$ rispetto alla base scelta;

(c) det. un compl. diretto di $L(A)$.

(a) $A \Rightarrow \alpha \begin{pmatrix} 1 & 0 \\ -2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}$

$$\Rightarrow L(A) = \left\{ \begin{pmatrix} \alpha & -2\beta \\ -2\alpha & 0 \end{pmatrix} \middle| \alpha, \beta \in \mathbb{R} \right\}$$

$$B_{L(A)} = \left(\begin{pmatrix} 1 & 0 \\ -2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix} \right) \quad \dim L(A) = 2$$

(b) $\begin{pmatrix} 2 & 5 \\ -4 & 0 \end{pmatrix} = \alpha \begin{pmatrix} 1 & 0 \\ -2 & 0 \end{pmatrix} + \beta \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \alpha & -2\beta \\ -2\alpha & 0 \end{pmatrix}$

$$\Rightarrow \alpha = 2 \quad \beta = -5/2$$

$$(2, -5/2)$$

Ⓒ $\dim M_2(\mathbb{R}) = 4$ $\left(B_{M_2(\mathbb{R})} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \dots \right)$

$$C = \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right)$$

- $\left(\begin{pmatrix} 1 & 0 \\ -2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right)$

$$\begin{pmatrix} \alpha+\delta & -2\beta \\ -2\alpha & \gamma \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{matrix} \delta = 0 & \beta = 0 \\ \alpha = 0 & \gamma = 0 \end{matrix}$$

$\Rightarrow$ è libero

$$\underset{\dim 2}{L(C)} + \underset{\dim 2}{L(A)} = \underset{\dim 4}{M_2(\mathbb{R})}$$

$$\dim\left(L(C) \cap L(A)\right) = -4 + 2 + 2 = 0$$

$$L(C) \cap L(A) = \underline{0} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

$$L(C) \oplus L(A)$$

ES2: $A = ((h, h, 2h+1, h), (h, h, 2h, h), (h, 0, h-1, h))$

per quali $h \in \mathbb{R}$, A è LEGATO?

- A è legato $\iff \rho(B) < 3$ (B: matrice delle compon.)

$$\rho(B) = \rho\begin{pmatrix} h & h & h \\ h & h & 0 \\ 2h+1 & 2h & h-1 \\ h & h & h \end{pmatrix} \qquad \underline{1^a R = 4^a R}$$

$$\det\begin{pmatrix} h & h & h \\ h & h & 0 \\ 2h+1 & 2h & h-1 \end{pmatrix} = h^2 \det\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 2h+1 & 2h & h-1 \end{pmatrix} =$$

$$= h^2 \det\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 2h+1 & 2h & h-1 \end{pmatrix} = h^2 \cdot (1) \cdot (-1)^{1+3} \cdot \det\begin{pmatrix} 1 & 1 \\ 2h+1 & 2h \end{pmatrix} =$$

$$= h^2 \cdot (2h - 2h - 1) = -h^2 .$$

A è LEGATA $\iff h = 0$.

ES.3: $A = ((1,2,2-k,2),(1,1-k,0,k-1),(0,2-k,2-k,0))$

per quali $k \in \mathbb{R}$ A è LIBERA?

- A è LIBERA $\iff \rho(B) = 3$

$$B = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1-k & 2-k \\ 2-k & 0 & 2-k \\ 2 & k-1 & 0 \end{pmatrix}$$

$$\det\begin{pmatrix} 1 & 1 & 0 \\ 2 & 1-k & 2-k \\ 2-k & 0 & 2-k \end{pmatrix} = 1 \cdot (-1)^{1+1} \det\begin{pmatrix} 1-k & 2-k \\ 0 & 2-k \end{pmatrix} +$$

$$+ 1 \cdot (-1)^{1+2} \det\begin{pmatrix} 2 & 2-k \\ 2-k & 2-k \end{pmatrix} =$$

$$= \ldots = 2k^2 - 5k + 2 \to k = 2, \quad k = \frac{1}{2}$$

- se $k \neq 2 \wedge k \neq \frac{1}{2} \Rightarrow \rho(B) = 3 \Rightarrow$ A è LIBERO

· se $K=2$ $\quad B=\begin{pmatrix} 1 & 1 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 0 \\ 2 & 1 & 0 \end{pmatrix}$ $\quad \rho(B)=2$

$\Rightarrow$ A è LEGATO.

· se $K=\frac{1}{2}$ $\quad B=\begin{pmatrix} 1 & 1 & 0 \\ 2 & 1/2 & 3/2 \\ 3/2 & 0 & 3/2 \\ 2 & -1/2 & 0 \end{pmatrix}$ $\quad M_2=\begin{pmatrix} 1 & 1 & 0 \\ 3/2 & 0 & 3/2 \\ 2 & -1/2 & 0 \end{pmatrix}$

$\det M_2 \neq 0$

$\Rightarrow \rho(B)=3 \Rightarrow$ A è LIBERO

A è LIBERO $\Leftrightarrow K\neq 2$.

ES.4: $\mathbb{R}^4(\mathbb{R})$

$A=\{(1,0,2,3),(0,1,0,1),(1,1,2,4)\}$

$B=\{(0,0,2,k),(3,2,3,1)\}$.

Per quali $k\in\mathbb{R}$ $\quad L(A)\oplus L(B)$.

$$A^* = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 0 & 2 \\ 3 & 1 & a \end{pmatrix} \qquad 1^\circ C + 2^\circ C = 3^\circ C$$

$\rho(A^*) = 2 \quad \left(\det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \neq 0\right)$

$B_{L(A)} = \left((1,0,2,3),(0,1,0,1)\right)$

$\dim L(A) = 2.$

$$B^* = \begin{pmatrix} 0 & 3 \\ 0 & 2 \\ 2 & 3 \\ k & 1 \end{pmatrix} \qquad \det\begin{pmatrix} 0 & 2 \\ 2 & 3 \end{pmatrix} \neq 0$$

$\rho(B^*) = 2 \quad \forall k \in \mathbb{R}$

$L(B) = 2 \quad B_{L(B)} = \left((0,0,2,k),(3,2,3,1)\right)$

$L(B) \oplus L(A): \left((1,0,2,3),(0,1,0,1),(0,0,2,k),(3,2,3,1)\right)$

$$C = \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 2 & 0 & 2 & 3 \\ 3 & 1 & k & 1 \end{pmatrix} \Rightarrow \underline{\underline{\rho(C) = 4}}$$

$$\det C \neq 0 \Leftrightarrow \rho(C) = 4$$

$$\det C = \det \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 2 & 0 & 2 & -3 \\ 3 & 1 & k & -8 \end{pmatrix} = 1 \cdot (-1)^{1+1} \cdot \det \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & -3 \\ 1 & k & -8 \end{pmatrix} =$$

$$= 1 \cdot (-1)^{1+1} \cdot \det \begin{pmatrix} 2 & -3 \\ k & -8 \end{pmatrix} + 1 (-1)^{3+1} \cdot \det \begin{pmatrix} 0 & 2 \\ 2 & -3 \end{pmatrix} =$$

$$= 3k - 20 \qquad \text{se } k \neq \frac{20}{3} \Rightarrow \rho(C) = 4$$

$$\Rightarrow L(A) + L(B) = \mathbb{R}^4$$

F. di Grassmann $\dim (L(A) \cap L(B)) = 0$

$$\Rightarrow L(A) \cap L(B) = \underline{0} = \{(0,0,0,0)\}$$

$$k \neq \frac{20}{3} \Rightarrow L(A) \oplus L(B)$$

ES.5 $A = \left(\begin{pmatrix} h & h \\ 2h+1 & h \end{pmatrix}, \begin{pmatrix} h & h \\ 2h & h \end{pmatrix}, \begin{pmatrix} h & 0 \\ h-1 & h \end{pmatrix} \right)$

per quali $h \in \mathbb{R}$, A è LEGATA?

- le componenti rispetto alla base canonica di $M_2(\mathbb{R})$. $(h, h, 2h+1, h), \dots$

- $C = \begin{pmatrix} h & h & h \\ h & h & 0 \\ 2h+1 & 2h & h-1 \\ h & h & h \end{pmatrix}$ (es.2)

$h = 0 \iff A$ è legata.

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$\boxed{\text{ES.2}}$. In $M_2(\mathbb{R})$

$$U = \left\{ \begin{pmatrix} \alpha+\gamma & \beta+\gamma \\ \alpha+\beta+2\gamma & -\beta-\gamma \end{pmatrix} \in M_2(\mathbb{R}) \;\middle|\; \alpha, \beta, \gamma \in \mathbb{R} \right\}$$

$$A = \begin{pmatrix} 2K & 1 \\ -1 & 2K+1 \end{pmatrix} \quad K \in \mathbb{R}.$$

ⓐ una base e la dim. U.

ⓑ per quali $K \in \mathbb{R}$ $A \in U$.

ⓐ $\begin{pmatrix} \alpha+\gamma & \beta+\gamma \\ \alpha+\beta+2\gamma & -\beta-\gamma \end{pmatrix} = \alpha \underbrace{\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}}_{U_1} + \beta \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}}_{U_2} + \gamma \underbrace{\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}}_{U_3}$

$U_1 + U_2 = U_3$ (U_1, U_2, U_3) è ins. LEGATO

$\Rightarrow B_U = (U_1, U_2)$ $\dim U = 2$

ⓑ $A = \begin{pmatrix} 2K & 1 \\ -1 & 2K+1 \end{pmatrix}$

$A \in U \iff \rho(C) = 2$

$C = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 2K & 1 & -1 & 2K+1 \end{pmatrix}$ $4^a C = 1^a C - 3^a C$

$$\det\begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ 2k & 1 & -1\end{pmatrix} = 0 \quad \left(\Rightarrow \rho(C)=2\right)$$

$\downarrow C^*$

$$\det C^* = -1-2k-1 = -2-2k$$

$$\rho(C)=2 \iff -2-2k=0 \iff k=-1$$

$$\Rightarrow A \in U$$

$\boxed{\text{ES. 3}}$: In $\mathbb{R}^4(\mathbb{R})$

$$W = L\left((1,1,4,0),(0,0,1,0)\right)$$

$$A_k = \left[(1,-1,2,-2),(k,0,2k,1),(0,1,k+1,0)\right]$$

ⓐ al variare di $k \in \mathbb{R}$: $\dim L(A_k)$ e $B_{L(A_k)}$

ⓑ $k=0$: una base e la dim. di $L(A_0)+W$ e $L(A_0)\cap W$.

ⓐ $C = \begin{pmatrix} 1 & -1 & 2 & -2 \\ k & 0 & 2k & 1 \\ 0 & 1 & k+1 & 0 \end{pmatrix}$

$M_1 = \begin{pmatrix} 1 & -1 & 2 \\ k & 0 & 2k \\ 0 & 1 & k+1 \end{pmatrix}$ $\det M_1 = k \cdot (k+1)$

. $k \neq 0 \wedge k \neq -1 \Rightarrow \rho(C) = 3$

. $k = 0$ $C = \begin{pmatrix} 1 & -1 & 2 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & \underbrace{1 & 1 & 0} \end{pmatrix}$ $\rho(C) = 3$

. $k = -1$ $C = \begin{pmatrix} 1 & -1 & 2 & -2 \\ -1 & 0 & -2 & 1 \\ 0 & \underbrace{1 & 0 & 0} \end{pmatrix}$ $\rho(C) = 3$

$\rightarrow \forall k \in \mathbb{R}$ $\rho(C) = 3 \Rightarrow \dim L(A_k) = 3$

$B_{L(A_k)} = A_k$

ⓑ $k=0$ $\qquad \underbrace{L(A_0)+W} \qquad L(A_0)\cap W$

W: $\dim W = 2$

- A_0: $(1,-1,2,-2),(0,0,0,1),(0,1,1,0)$

$B_{L(A_0)} = ((\ldots),(\ldots),(\ldots))$

$\dim L(A_0) = 3$

- $\dim(L(A_0)+W)$: $\{$

$$C = \begin{pmatrix} 1 & -1 & 2 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$\boxed{\rho(C) = 4}$ perché: $\det\begin{pmatrix} 1 & -1 & 2 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} =$

$$= 1\cdot(-1)^{2+4}\det\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = 1\cdot(-1)^{3+3}\det\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = 1$$

$$\dim\left(L(A_0)+W\right)=4$$

$$\mathcal{B}_{L(A_0)+W} = \text{Base canonica di } \mathbb{R}^4$$

- $L(A_0)\cap W$:

$$\dim\left(L(A_0)\cap W\right)=\dim\left(L(A_0)\right)+\dim W-\dim\left(L(A_0)+W\right)=$$
$$=3+2-4=1$$

$$\alpha(1,-1,2,-2)+\beta(0,0,0,1)+\gamma(0,1,0,1)=$$
$$=x(1,1,4,0)+y(0,0,1,0)$$

$$(\alpha,-\alpha+\gamma,2\alpha,-2\alpha+\beta+\gamma)=(x,x,4x+y,0)$$

$$\ldots\quad L(A_0)\cap W=\left\{(x,x,4x,0)\,|\,x\in\mathbb{R}\right\}$$

$$\Rightarrow \dim\left(L(A_0)\cap W\right)=1 \quad \mathcal{B}_{L(A_0)\cap W}=\left((1,1,4,0)\right).$$

Esercizi da svolgere

1) In uno spazio vettoriale V di dimensione 4 su $\mathbb{R}$, è assegnata una base $B=\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_4\}$ e due insiemi $A=\{\mathbf{e}_1-\mathbf{e}_2+2\mathbf{e}_3+\mathbf{e}_4,\mathbf{e}_2+\mathbf{e}_3\}$, $D=\{\mathbf{e}_2,\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4\}$; si ricerchi:

a) $L(A)$, $L(D)$; b) $L(A)\cap L(D)$; c) $L(A)+L(D)$;

d) Esiste un s.s.v. W tale che $L(A)+W = V$?

2) Trovare la dimensione e una base per $V+W$ in $\mathbb{R}^4(\mathbb{R})$ dove :

$$V=\{ (x,y,z,t)\in \mathbb{R}^4 \mid y=0 , t-x=0\}$$

$$W = \{ (x,y,z,t) \in \mathbb{R}^4 \mid z-t=x-y=0\}.$$

3) Dati i sottospazi vettoriali di $\mathbb{R}^4(\mathbb{R})$ U e W determinare una base per $U\cap W$, $U+W$:

$$U=\{(0,-a,a+2b,b) \mid a,b\in\mathbb{R}\}$$

$$W=\{(0,-2x,x-2y,3x) \mid x, y\in\mathbb{R}\}.$$

4) Determinare la dimensione delle coperture lineari generate dai seguenti insiemi e trovare una base di tali sottospazi:

a) $A_1=\{(3,-1,-1),(3,0,-3),(1,-2,3),(5,-1,-3)\}$;

b) $A_2=\{(1,0,1,-1),(1,-1,0,-1),(2,-3,3,-2),(1,0,1,1)\}$.

5) Per quali valori di h, numero reale, la sequenza $A=((h,h,2h+1,h), (h,h,2h,h), (h,0,h-1,h))$ è legata?

6) Determinare per quali valori di k la sequenza A è libera, dove $A=((1,2,2-k,2), (1,1-k,0,k-1), (0,2-k,2-k,0))$.

7) Dati i sottoinsiemi:

$A=\{(1,0,2,3),(0,1,0,1),(1,1,2,4)\}$ e $B=\{(0,0,2,k),(3,2,3,1)\}$

dello spazio vettoriale $\mathbb{R}^4(\mathbb{R})$, si trovi per quali valori reali di k, $\mathbb{R}^4(\mathbb{R}) = L(A) \oplus L(B)$.