<u>ES</u>: In $\mathbb{R}^4(\mathbb{R})$. Per quali $K \in \mathbb{R}$
$v = (0, K-1, K-1, 2) \in$ allo spazio vett. generato
da $A = ((0, K-1, K-1, K-1), (0, 0, K-2, 2K-4)$
$(0, 0, 0, 2K-4))$.

- $L(A)$: $\dim L(A) = \rho(B)$

$$B = \begin{pmatrix} 0 & K-1 & K-1 & K-1 \\ 0 & 0 & K-2 & 2K-4 \\ 0 & 0 & 0 & 2K-4 \end{pmatrix} \qquad \rho(B) \leq 3$$

Ⓐ • $K \neq 1$ e $K \neq 2$ $\Rightarrow \rho(B) = 3$

- A è base di $L(A)$

Ⓑ • $k = 1$

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

$\det \begin{pmatrix} -1 & -2 \\ 0 & -2 \end{pmatrix} \neq 0 \qquad \rho(B) = 2$

- Base: $((0, 0, -1, -2), (0, 0, 0, -2))$

ⓒ. $K=2$ $\quad B:\begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\rho(B)=1$ Base $((0,1,1,1))$

Ⓐ consideriamo

$$C=\begin{pmatrix} 0 & k-1 & k-1 & k-1 \\ 0 & 0 & k-2 & 2k-4 \\ 0 & 0 & 0 & 2k-4 \\ 0 & k-1 & k-1 & 2 \end{pmatrix}$$

$v \in L(A) \iff \rho(C)=3 \iff$

$$\rho\begin{pmatrix} k-1 & k-1 & k-1 \\ 0 & k-2 & 2(k-2) \\ 0 & 0 & 2(k-2) \\ k-1 & k-1 & 2 \end{pmatrix}=3 \qquad v\in L(A)$$

Ⓑ $\quad v\in L(A) \iff \rho\begin{pmatrix} 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 2 \end{pmatrix}=2$

ⓒ $\quad v\in L(A) \iff \rho\begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix}=1$

ma $M=\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \to \det M \neq 0 \Rightarrow$ il rango è 2.

se $K \neq 2$ $\quad v \in L(A)$

se $K = 2$ $\quad v \notin L(A)$

Siano U, W sottospazi vett. di $V(\mathbb{K})$

Def. 2 operazioni:

$$U+W = \{u+w \in V \mid u \in U, w \in W\}$$

$$U \cap W = \{v \in V \mid v \in U \wedge v \in W\}$$

- $U+W$, $U \cap W$ sono sottospaz. vett. di $V(\mathbb{K})$.

- FORMULA DI GRASSMANN:

$$\dim(U+W) = \dim U + \dim W - \dim(U \cap W)$$

ES1: Su $\mathbb{R}^3$

$U=\{(\alpha,0,-\beta)\mid \alpha,\beta\in\mathbb{R}\}$ $W=\{(0,3\gamma,\delta)\mid \gamma,\delta\in\mathbb{R}\}$

det. $U+W$, $U\cap W$.

- B_U: $(\alpha,0,-\beta)=\alpha(1,0,0)+\beta(0,0,-1)$

  $B_U=((1,0,0),(0,0,-1))$ $\dim U=2$

- B_W: $(0,3\gamma,\delta)=\gamma(0,3,0)+\delta(0,0,1)$

  $B_W=((0,3,0),(0,0,1))$ $\dim W=2$

- $U\cap W$: $\alpha,\beta,\gamma,\delta\in\mathbb{R}$

  $(\alpha,0,-\beta)=(0,3\gamma,\delta)$ $\begin{cases}\alpha=0\\ \gamma=0\\ \beta=-\delta\end{cases}$

  $U\cap W:=\{(0,0,\delta)\mid \delta\in\mathbb{R}\}$

  $B_{U\cap W}=((0,0,1))$ $\dim U\cap W=1$

- F. di Grassmann: $\dim U+W=\dim U+\dim W-\dim U\cap W$

  $=2+2-1=3$

- $B_{U+W}: ((0,0,1),(0,1,0),(1,0,0))$.

ES2: $\mathbb{R}^4(\mathbb{R})$

$U = \{(\alpha, 0, -\beta, 3\alpha) \mid \alpha, \beta \in \mathbb{R}\}$ e $W = \{(0, 2\gamma, \delta, 0) \mid \gamma, \delta \in \mathbb{R}\}$

det. $U+W$, $U \cap W$.

- $B_U = ((1,0,0,3),(0,0,-1,0))$ $\dim U = 2$
- $B_W = ((0,2,0,0),(0,0,1,0))$ $\dim W = 2$
- $U \cap W$: $\alpha, \beta, \gamma, \delta \in \mathbb{R}$:

$$(\alpha, 0, -\beta, 3\alpha) = (0, 2\gamma, \delta, 0)$$

$$\begin{cases} \alpha = 0 \\ \gamma = 0 \\ \beta = -\delta \end{cases}$$

$U \cap W = \{(0,0,\delta,0) \mid \delta \in \mathbb{R}\}$

$B_{U \cap W} = ((0,0,1,0))$ $\dim U \cap W = 1$

F. di Grassmann: $\dim U+W = 3$

insieme di generatori di $U+W$:

$\{(1,0,0,3),(0,2,0,0),$
$(0,0,-1,0),(0,0,1,0)\}$ è LEGATO

$B_{U+W} = ((1,0,0,3),(0,2,0,0),(0,0,1,0))$

i vettori di B_{U+W} infatti sono lin. indip.

perchè $\rho\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \\ 3 & 0 & 0 \end{pmatrix} = 3$

$U+W = \{(\alpha, 2\beta, \gamma, 3\alpha) \mid \alpha, \beta, \gamma \in \mathbb{R}\}$

SOMMA DIRETTA

Siano U, W due sottospazi vett. di $V(\mathbb{K})$.

$V = U \oplus W$ SSE $U+W = V$
$U \cap W = \{\underline{0}\}$

$\downarrow$

<u>oss</u>: $\exists!\, u \in U$ e $!\, w \in W$, $\forall v \in V$: $v = u + w$

ES.3: Verificare se U e W sono uno complemento diretto dell'altro.

In $\mathbb{R}^3$: $U = \{(\alpha, 2\alpha, \beta) \mid \alpha, \beta \in \mathbb{R}\}$

$W = \{(\gamma, \delta, 2\delta) \mid \gamma, \delta \in \mathbb{R}\}$

- $B_U = ((1,2,0),(0,0,1)) \rightarrow \dim U = 2$
- $B_W = ((1,0,0),(0,1,2)) \rightarrow \dim W = 2$

$U \cap W$: $\alpha, \beta, \gamma, \delta \in \mathbb{R}$

$$(\alpha, 2\alpha, \beta) = (\gamma, \delta, 2\delta) \qquad \begin{cases} \alpha = \gamma \\ \delta = 2\gamma \\ \beta = 4\gamma \end{cases}$$

$U \cap W = \{(\gamma, 2\gamma, 4\gamma) \mid \gamma \in \mathbb{R}\}$

$B_{U \cap W} = ((1,2,4)) \quad \dim U \cap W = 1$

$\Rightarrow U \not\oplus W$

$$\left(U+W: \quad \dim U+W = 2+2-1=3 \right.$$
$$\left. U+W = \mathbb{R}^3 . \right)$$

l'unico sottospazio di $\mathbb{R}^3$ di dim. 3 è $\mathbb{R}^3$ stesso.

ES. 4: Considerando U (def. ES. 2), determinare $T \subseteq \mathbb{R}^4$: $T \oplus U = \mathbb{R}^4$

- $U = \{(\alpha, 0, -\beta, 3\alpha) \mid \alpha, \beta \in \mathbb{R}\}$ $\quad \dim U = 2$
- $\dim T = 2$ $\quad$ (F. d. Grassman: $\dim T = \dim(U+T) + \dim(U \cap T) - \dim U$)

$\downarrow$

$B_T : ((0,1,0,0),(0,0,0,1))$

$T = \{(0, x, 0, y) \mid x, y \in \mathbb{R}\}$

$\dim T = 2$

$T+U \rightarrow \quad A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 3 & 0 & 0 & 1 \end{pmatrix} \quad \rho(A) = 4 \Leftrightarrow \dim(T+U) = 4$

- $\det A \neq 0 \quad \det A = 1 \cdot (-1)^{1+1} \cdot \det \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1$

$\rho(A) = 4$

$T+U = \mathbb{R}^4$

$T \oplus U = \mathbb{R}^4.$

b) $v = (1,1,1,1)$. $t \in T \; u \in U : v = t + u$

$(1,1,1,1) = \underbrace{\alpha(1,0,0,3) + \beta(0,0,-1,0)}_{u} + \underbrace{\gamma(0,1,0,0) + \delta(0,0,0,1)}_{t}$

$(1,1,1,1) = (\alpha, 0, -\beta, 3\alpha) + (0, \gamma, 0, \delta)$

$(\alpha, \gamma, -\beta, \delta + 3\alpha) \rightarrow \begin{array}{l} \alpha = 1 \\ \gamma = 1 \quad \beta = -1 \\ \delta = -2 \end{array}$

$$u = (\alpha, 0, -\beta, 3\alpha) \underset{\uparrow}{=} (1, 0, 1, 3) \in U$$

$\alpha = 1$

$\beta = -1$

$$t = (0, \gamma, 0, \delta) \underset{\uparrow}{=} (0, 1, 0, -2) \in T$$

$\gamma = 1$

$\delta = -2$

t e u sono unici perché $U \oplus T = \mathbb{R}^4$.

ES5: Dati

$V = \{(x, y, z, t) \mid y = 0 \wedge t + x = 0\}$

$W = \{(x, y, z, t) \mid z = t = x - y = 0\}$

ⓐ la dim. e una base per $V + W$ (in $\mathbb{R}^4$).

ⓑ la dim. e una base per $V \cap W$ (in $\mathbb{R}^4$).

ⓒ $V \oplus W$?

TRACCIA : $\cdot\, B_V = \big((1,0,0,-1),(0,0,1,0)\big)$

$\dim V = 2$

$\cdot\, B_W = \big((1,1,0,0)\big) \quad \dim W = 1$

$\cdot\, V \cap W : \ (\alpha, 0, \beta, -\alpha) = (x, x, 0, 0)$

$\alpha = \beta = x = 0$

$\dim(V \cap W) = 0$

F. di G. $\dim(V+W) = \dim V + \dim V - \dim(V \cap W)$

$= 2 + 1 - 0 = 3$

$B_{V+W} : \dots \qquad V \not\oplus W$

<u>ES.6</u>: $A = \{(-1,1,0),(0,2,1)\}$

$B = \{(-2,1,3),(0,-2,0)\}$.

Dopo aver studiato i sottospt. vett. di $\mathbb{R}^3$, $L(A), L(B)$, determinare una base per $L(A) \cap L(B)$, $L(A) + L(B)$.

$L(A) \oplus L(B)$?

$\cdot\, L(A)$: $B_{L(A)} = \big((-1,1,0),(0,2,1)\big) \quad \dim L(A) = 2$

$L(A) = \{(-\alpha, \alpha + 2\beta, \beta) \mid \alpha, \beta \in \mathbb{R}\}$

- $L(B)$: $B_{L(B)} = ((-2,1,3),(0,-2,0))$

  $\dim L(B) = 2$ $L(B) = \{(-2\gamma, \gamma-2\delta, 3\gamma) \mid \gamma,\delta \in \mathbb{R}\}$

- $L(A) \cap L(B)$: $(-\alpha, \alpha+2\beta, \beta) = (-2\gamma, \gamma-2\delta, 3\gamma)$

  $\Rightarrow \alpha = 2\gamma$ $\delta = -\frac{7}{2}\gamma$

  $\beta = 3\gamma$

  $L(A) \cap L(B) = \{(-2\gamma, 8\gamma, 3\gamma) \mid \gamma \in \mathbb{R}\}$

  $B_{L(A)\cap L(B)} = ((-2,8,3))$

  $\dim(L(A)\cap L(B)) = 1$

- $L(A)+L(B)$: $\left.\begin{matrix}(-1,1,0),(0,2,1)\\(-2,1,3),(0,-2,0)\end{matrix}\right\}$ 4 vett. in $\mathbb{R}^3$

  $\downarrow$

  $((-1,1,0),(0,2,1),(-2,1,3))$

  $\uparrow$ sono lin. ind.

  $B_{L(A)+L(B)}$. $\dim(L(A)+L(B)) = 3$

- $L(A) \not\oplus L(B)$

ES 7: Su $V(\mathbb{K})$ spz. vett. $\dim V = 4$.

$B_1 = (e_1, e_2, e_3, e_4)$.

(a) Verificare che $B_2 = (e_1 + 2e_2, 2e_2, e_3 + e_4, e_3)$ è base di V.

(b) det le componenti di $w = e_1 + 8e_2 + e_3 + 2e_4$ rispetto a B_2.

(c) det un COMPL. DIRETTO di

$$U = \{2xe_1 + xe_2 + 3xe_4 \in V \mid x \in \mathbb{R}\}$$

(a)

- B_1 è base $\Rightarrow$ · ogni vett. di V è COMB. LIN. (e_1, e_2, e_3, e_4)
  · B_1 è LIBERO

B_2 è LIBERO: $\alpha, \beta, \gamma, \delta \in \mathbb{R}$

$$\alpha(e_1 + 2e_2) + \beta(2e_2) + \gamma(e_3 + e_4) + \delta(e_3) = \underline{0}$$

$$\alpha e_1 + 2\alpha e_2 + 2\beta e_2 + \gamma e_3 + \gamma e_4 + \delta e_3 = \underline{0}$$

$$e_1(\alpha) + e_2(2\alpha + 2\beta) + e_3(\gamma + \delta) + e_4(\gamma) = \underline{0}$$

$\alpha = 0 \quad 2\alpha + 2\beta = 0 \quad \gamma + \delta = 0 \quad \gamma = 0$ (perché B_1 è LIBERO)

$\beta = 0 \quad \delta = 0$

$\Rightarrow B_2$ è LIBERO. $\Rightarrow B_2$ è base per V.

oppure, per dim. che B_2 è base, possiamo usare le componenti di B_2 rispetto alla base B_1 e verificare che la matrice di tali componenti ha rango 4.

$$\rho\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = 4 \Rightarrow$$

i 4 vett. che generano B_2 sono lin. indip. $\Rightarrow$ generano V ($\dim V = 4$).

b) $w = e_1 + 8e_2 + e_3 + 2e_4 \; : (1, 8, 1, 2)$

$$(1,8,1,2) = \alpha(1,0,0,0) + \beta(2,2,0,0) + \gamma(0,0,1,1) + \delta(0,0,1,0)$$

$\alpha = 1 \quad \beta = 3 \quad \gamma = 2 \quad \delta = -1$

$\alpha = 1 \quad \beta = 3 \quad \gamma = 2 \quad \delta = -1$

$w_{B_2} = (1, 3, 2, -1)$.