$$G = \begin{pmatrix} 0 & 4 & 1 & 2 & 2 \\ 0 & 1 & 4 & 5 & 2 \\ -2 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & -2 & 0 \\ 2 & 2 & 1 & 0 & -1 \end{pmatrix} \in M_5(\mathbb{R})$$

$$\det G = g_{41} \cdot M_{41} + g_{42} M_{42} + g_{43} M_{43} + g_{44} M_{44} + g_{45} M_{45}$$

$$= 1 \cdot M_{41} - 2 M_{44}$$

$$M_{41} = (-1)^{4+1} \det \begin{pmatrix} 4 & 1 & 2 & 2 \\ 1 & 4 & 5 & 2 \\ 0 & 0 & 1 & 1 \\ 2 & 1 & 0 & -1 \end{pmatrix} =$$

$$= -1 \cdot \left(1 \cdot (-1)^{3+3} \det \begin{pmatrix} 4 & 1 & 2 \\ 1 & 4 & 2 \\ 2 & 1 & -1 \end{pmatrix} + 1 (-1)^{3+4} \det \begin{pmatrix} 4 & 1 & 2 \\ 1 & 4 & 5 \\ 2 & 1 & 0 \end{pmatrix} \right) =$$

$$= -1 \left(-16 + 4 + 2 - 16 - 8 + 1 - 1 \cdot (10 + 2 - 16 - 20) \right) = 9$$

$$M_{44} = (-1)^{4+4} \det \begin{pmatrix} 0 & 4 & 1 & 2 \\ 0 & 1 & 4 & 2 \\ -2 & 0 & 0 & 1 \\ 2 & 2 & 1 & -1 \end{pmatrix} =$$

$$= -2 \cdot (-1)^{3+1} \det \begin{pmatrix} 4 & 1 & 2 \\ 1 & 4 & 2 \\ 2 & 1 & -1 \end{pmatrix} + 2 (-1)^{4+1} \det \begin{pmatrix} 4 & 1 & 2 \\ 1 & 4 & 2 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= -2 (-16 + 4 + 2 - 16 - 8 - 1) - 2 (16 - 1) = 36$$

$$\det G = 1 \cdot M_{41} - 2 M_{44} = 1 \cdot 9 - 2 \cdot 36 = -63$$

TRASFORMAZIONI ELEMENTARI

$A \in \mathbb{K}^{m,n}$

T_1: scambiare 2 RIGHE (2 COLONNE) di A

T_2: sommare ad una RIGA (colonna) di A il prodotto di un'altra riga (colonna) di A per uno scalare.

T_3: moltiplicare una riga (colonna) di A per uno scalare $\lambda \in \mathbb{K}$.

$A \in M_n(\mathbb{K})$ e $B \in M_n(\mathbb{K})$ ott. da A

1) T_1: $\det B = -\det A$

2) T_2: $\det B = \det A$

3) T_3: $\det B = \lambda \cdot \det A$

ESERCIZIO:

$$A=\begin{pmatrix} 1 & 0 & -4 & 6 \\ 4 & 4 & 1 & 2 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} \in M_4(\mathbb{R})$$

$$\det A \underset{T_2}{=} \det\begin{pmatrix} 1 & 0 & -4 & 6 \\ 0 & 4 & 17 & -22 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} \underset{T_2}{=} \det\begin{pmatrix} 1 & 4 & -4 & 6 \\ 0 & -13 & 17 & -22 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} =$$

$2^a R = 2^a R - 4 \cdot 1^a R$ $\qquad$ $2^a C = 2^a C - 3^a C$

$$= 1 \cdot (-13) \cdot (-2) \cdot (-3) = -78$$

ES. 2:

$$B=\begin{pmatrix} 0 & 4 & 1 & 2 & 2 \\ 0 & 1 & 4 & 5 & 2 \\ -4 & 0 & 0 & 1 & 1 \\ 2 & 0 & 0 & -2 & 0 \\ 4 & 2 & 1 & 0 & -1 \end{pmatrix} \in M_5(\mathbb{R})$$

$$\det B \underset{T_3}{=} 2 \cdot \det\begin{pmatrix} 0 & 4 & 1 & 2 & 2 \\ 0 & 1 & 4 & 5 & 2 \\ -2 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & -2 & 0 \\ 2 & 2 & 1 & 0 & -1 \end{pmatrix} \underset{T_2}{=} 2 \cdot \det\begin{pmatrix} 0 & 4 & 1 & 2 & 2 \\ 0 & 1 & 4 & 5 & 2 \\ -2 & 0 & 0 & -3 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 4 & -1 \end{pmatrix} =$$

$4^a C + 2 \cdot 1^a C$

$$= 2 \cdot (1) \cdot (-1)^{4+1} \det \begin{pmatrix} 4 & 1 & 2 & 2 \\ 1 & 4 & 5 & 2 \\ 0 & 0 & -3 & 1 \\ 2 & 1 & 4 & -1 \end{pmatrix} = -2 \cdot \det \begin{pmatrix} 4 & 1 & 8 & 2 \\ 1 & 4 & 11 & 2 \\ 0 & 0 & 0 & 1 \\ 2 & 1 & 1 & -1 \end{pmatrix} =$$

$$= -2\,(1) \cdot (-1)^{3+4} \det \begin{pmatrix} 4 & 1 & 8 \\ 1 & 4 & 11 \\ 2 & 1 & 1 \end{pmatrix} = 2 \cdot \det \begin{pmatrix} 2 & 1 & 7 \\ -7 & 4 & 7 \\ 0 & 1 & 0 \end{pmatrix} \underset{T_3}{=}$$

$$= 14 \cdot \det \begin{pmatrix} 2 & 1 & 1 \\ -7 & 4 & 1 \\ 0 & 1 & 0 \end{pmatrix} = 14 \cdot (1) \cdot (-1)^{3+2} \det \begin{pmatrix} 2 & 1 \\ -7 & 1 \end{pmatrix} = -14\,(2+7) = -126$$

DA SVOLGERE:

$$A = \begin{pmatrix} 5 & 10 & 0 & 0 \\ 1 & 3 & 0 & 1 \\ 0 & -3 & -5 & -10 \\ 0 & 4 & 0 & 5 \end{pmatrix} \quad (-25)$$

$$B = \begin{pmatrix} 10 & 9 & 1 & -1 & 2/5 \\ 0 & 1 & 0 & 0 & 4 \\ 6/5 & 6 & 21 & 0 & 2/3 \\ 21 & 3 & 0 & -1 & 2 \\ 21 & 4 & 0 & -1 & 6 \end{pmatrix} \quad (0)$$

ESERCIZIO: $k \in \mathbb{R}$

$$C = \begin{pmatrix} 9 & -k & 3 & 1 \\ 0 & -k & -3k & 0 \\ 1 & 0 & 0 & k^2 \\ 0 & -1 & k+5 & 0 \end{pmatrix} \in M_4(\mathbb{R})$$

$$\det C \underset{T_3}{=} -k \det \begin{pmatrix} 9 & -k & 3 & 1 \\ 0 & +1 & +3 & 0 \\ 1 & 0 & 0 & k^2 \\ 0 & -1 & k+5 & 0 \end{pmatrix} \underset{T_1}{=} +k \det \begin{pmatrix} 1 & 0 & 0 & k^2 \\ 0 & 1 & 3 & 0 \\ \underline{9} & -k & 3 & 1 \\ 0 & -1 & k+5 & 0 \end{pmatrix} =$$

$$\underset{T_2}{=} k \det \begin{pmatrix} 1 & 0 & 0 & k^2 \\ 0 & 1 & 3 & 0 \\ 0 & -k & 3 & 1-9k^2 \\ 0 & -1 & k+5 & 0 \end{pmatrix} \underset{T_2}{=} k \cdot \det \begin{pmatrix} 1 & 0 & 0 & k^2 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 3+3k & 1-9k^2 \\ 0 & 0 & k+8 & 0 \end{pmatrix} \underset{T_1}{=}$$

$$= -k \det \begin{pmatrix} 1 & 0 & k^2 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1-9k^2 & 3+3k \\ 0 & 0 & 0 & k+8 \end{pmatrix} = -k \cdot (1-9k^2)(k+8)$$

$$k \neq 0 \ \vee\ k \neq \pm \frac{1}{3} \ \vee\ k \neq -8$$

Sia $A \in M_n(\mathbb{K})$.

TEOR. DELLA TRASPOSTA: $\det A = \det {}^tA$

TEOR. DI BINET: $\det(A \cdot B) = \det A \cdot \det B$
Sia $B \in M_n(\mathbb{K})$

SECONDO TEOR. di LAPLACE:

$$\sum_{j=1}^{m} a_{kj} \cdot M_{ij} = 0 \qquad k \neq i$$

$$\sum_{i=1}^{n} a_{i,j} \cdot M_{i,k} = 0 \qquad j \neq k$$

ESEMPIO: $A = \begin{pmatrix} 0 & 1 & -5 \\ 2 & 0 & 2 \\ 0 & -1 & 3 \end{pmatrix} \in M_3(\mathbb{R})$

$$a_{12} M_{13} + a_{22} M_{23} + a_{32} M_{33} = 0 \; ?$$

$$\left.\begin{aligned} M_{13} &= (-1)^{1+3} \det \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} = -2 \\ M_{23} &= (-1)^{5} \det \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} = 0 \\ M_{33} &= (-1)^{6} \det \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} = -2 \end{aligned}\right\} \; 1(-2) + 0 \cdot 0 - 1(-2) = 0$$

INVERSA: $A \in M_n(\mathbb{K})$. A è INVERTIBILE se $\exists B \in M_n(\mathbb{K})$ $(= A^{-1})$ t.c. : $A \cdot B = B \cdot A = I_n$

<u>TEOREMA</u>: $A \in M_n(\mathbb{K})$ è INVERTIBILE

$$\Updownarrow$$

$$\det A \neq 0.$$

$$A^{-1} = \frac{1}{\det A} \cdot \begin{pmatrix} M_{11} & M_{12} & \cdots & M_{1n} \\ \vdots & & & \\ M_{n1} & \cdots & & M_{nn} \end{pmatrix}^t$$

ESERCIZIO 1: $A = \begin{pmatrix} 1 & 2 \\ 4 & -3 \end{pmatrix} \in M_2(\mathbb{R})$

$$\det A = -11 \neq 0 \Rightarrow \exists A^{-1} \in M_2(\mathbb{R})$$

$$\left.\begin{aligned} M_{11} &= (-1)^2(-3) = -3 \\ M_{12} &= (-1)^3(4) = -4 \\ M_{21} &= (-1)^3(2) = -2 \\ M_{22} &= (-1)^4(1) = 1 \end{aligned}\right\} A^{-1} = -\frac{1}{11} \cdot \begin{pmatrix} -3 & -4 \\ -2 & 1 \end{pmatrix}^t =$$

$$= \begin{pmatrix} +\frac{3}{11} & \frac{2}{11} \\ \frac{4}{11} & -\frac{1}{11} \end{pmatrix}$$

ES 2: $B = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \in M_3(\mathbb{R})$

$$\det B = 1 \cdot (-1)^{3+3} \det \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} = -1 \neq 0 \Rightarrow \exists B^{-1} \in M_3(\mathbb{R})$$

$$M_{11} = (-1)^{1+1} \det \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = 1$$

$$M_{12} = (-1)^{1+2} \det \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} = -2$$

$$M_{13} = (-1)^{1+3} \det \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} = 0$$

$$M_{21} = (-1)^{2+1} \det \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = -1$$

$$M_{22} = (-1)^{2+2} \det \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = 1$$

$$M_{23} = (-1)^{2+3} \det \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = 0$$

$$M_{31} = (-1)^{3+1} \det \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 0$$

$$M_{32} = (-1)^{3+2} \det \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} = +1$$

$$M_{33} = (-1)^{3+3} \det \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} = -1$$

$$B^{-1} = \frac{1}{-1} \cdot \begin{pmatrix} 1 & -2 & 0 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}^t = -1 \begin{pmatrix} 1 & -1 & 0 \\ -2 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix} =$$

$$= \begin{pmatrix} -1 & 1 & 0 \\ 2 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

DA SVOLGERE: trovare, se esiste, l'inversa

$$A = \begin{pmatrix} -1 & 2 \\ 0 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$$

$$C = \begin{pmatrix} 2 & 2 & 0 \\ -1 & 1 & 0 \\ 0 & -3 & -5 \end{pmatrix} \qquad D = \begin{pmatrix} 3 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & -3 & -5 \end{pmatrix}$$

$$E = \begin{pmatrix} 0 & 1 & k-3 & -k \\ 4 & -1 & 2 & 2k \\ 0 & 1 & 0 & 0 \\ 7 & 3-k & -1 & 2 \end{pmatrix}$$