PROVA INTERMEDIA ① 15/11/'10

ES 1:

$$A=\begin{pmatrix} 1 & 1 & k+1 \\ 0 & k+3 & k \\ 2 & 3 & k+2 \end{pmatrix} \qquad B=\begin{pmatrix} k+1 \\ k+3 \\ -1 \end{pmatrix}$$

ⓐ $\rho(A)=?$

$\det A = -k\cdot(k+2)$ per $k\neq 0,-2$ $\rho(A)=3$

per $k=0 \wedge k=-2$ $\rho(A)=2$

ⓑ $k=0$

$$A|B=\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 3 & 0 & 3 \\ 2 & 3 & 2 & -1 \end{array}\right) \quad |M_{124}|\neq 0 \quad \rho(A|B)=3$$

$K=-2$

$\rho(A|B)=2$

ⓒ $k\neq 0$; $k\neq 0,-2$ il sist. è comp. ha una soluz.

$k=-2$ ∞^1 sol.

ⓓ $k=-2$

$$A|B=\left(\begin{array}{ccc|c} 1 & 1 & -1 & -1 \\ 0 & 1 & 2 & 1 \\ 2 & 3 & 0 & -1 \end{array}\right) \qquad \begin{cases} x+y-z=-1 \\ y+2z=1 \end{cases}$$

$$S=\{(-2+3\alpha,\, 1-2\alpha,\, \alpha)\in\mathbb{R}^3 \mid \alpha\in\mathbb{R}\}$$

ES.2: $\mathbb{R}^4(\mathbb{R})$ $\quad v=(k+2,3,k+1,k+1)$

$A=[(1,1,k,1),(0,1,0,-1),(k+1,1,0,k)]$ $\quad k\in\mathbb{R}$.

(a) $\dim L(A)$ $\quad B_{L(A)}$.

$$A'=\begin{pmatrix}1&1&k&1\\0&1&0&-1\\k+1&1&0&k\end{pmatrix} \qquad |M_{123}|=-k\cdot(k+1)$$

$$k=0 \quad \begin{pmatrix}1&1&0&1\\0&1&0&-1\\1&1&0&0\end{pmatrix} \quad \rho(A')=3$$

$$k=-1 \quad \begin{pmatrix}1&1&-1&1\\0&1&0&-1\\0&1&0&-1\end{pmatrix} \quad \rho(A')=2$$

- $k\neq-1$: $\dim L(A)=3$ $\quad B_{L(A)}=A$
- $k=-1$: $\dim L(A)=2$ $\quad B_{L(A)}=((1,1,-1,1)(0,1,0,-1))$

(b) se $k=-1$ $\quad \rho\begin{pmatrix}1&1&-1&1\\0&1&0&-1\\1&3&0&0\end{pmatrix}=3 \Rightarrow v\notin L(A)$

se $k\neq-1$

$$\frac{A'}{v}=\begin{pmatrix}1&1&k&1\\0&1&0&-1\\k+1&1&0&k\\k+2&3&k+1&k+1\end{pmatrix} \qquad \det\left(\frac{A'}{v}\right)=(1-k)(1+k)$$

$$\Downarrow$$

$v\in L(A)$ se $k=1$

(c) $A^\perp$: $(k=-1)$ $\qquad A^\perp=L(A)^\perp$

$$\begin{cases}(x,y,z,t)\cdot(1,1,-1,1)=0\\(x,y,z,t)\cdot(0,1,0,-1)=0\end{cases} \qquad \begin{cases}x=z-2t\\y=t\end{cases}$$

$$S=\{(z-2t,t,z,t)\mid z,t\in\mathbb{R}\}=L((1,0,1,0)(-2,1,0,1))$$

ES3.: In $M_2(\mathbb{R})$

$$U = \left\{ \begin{pmatrix} \alpha+\beta+2\delta & 2\beta+2\delta \\ \beta+\delta & -\alpha+\gamma \end{pmatrix} \middle| \alpha,\beta,\gamma,\delta \in \mathbb{R} \right\}$$

(a) $\dim U$ e B_U

$$\alpha\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{u_1} + \beta\underbrace{\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}}_{u_2} + \gamma\underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{u_3} + \delta\underbrace{\begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix}}_{u_4}$$

$$A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \cancel{2} & \cancel{2} & \cancel{1} & \cancel{0} \end{pmatrix} \qquad \det A = 0 \qquad \rho(A) = 3$$

$\dim U = 3$

$B_U = (u_1, u_2, u_3)$

(b) $W = L\left(\begin{pmatrix} k-1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right)$

$$W^r = \begin{pmatrix} \boxed{\begin{matrix} k-1 & 1 \\ 1 & 1 \end{matrix}} & \begin{matrix} 1 & 0 \\ 1 & 0 \end{matrix} \end{pmatrix} \qquad k-1 \neq 1$$

- $k \neq 2$ $\dim W = 2$
- $k = 2$ $\dim W = 1$

(c) $U+W$ è diretta per $k=2$.

$\dim M_2(\mathbb{R}) = 4$ $\dim U = 3$ $\dim W = 1$

$$\begin{matrix} B_U \\ C \end{matrix} \left[\begin{pmatrix} & & & \\ 1 & 1 & 1 & 0 \end{pmatrix} \right. \to \rho(4)$$

ES.4:
$$A_k = \begin{pmatrix} 0 & 1 & 0 \\ 0 & k+1 & 0 \\ 2 & k & 2 \end{pmatrix}$$

(a) pol. caratt.: $(k+1-\lambda)\cdot\lambda\cdot(\lambda-2)$

$\lambda_1=0$

$\lambda_2=2$

$\lambda_3=k+1$

- $k \neq -1, +1$ λ_i distinti: $a_{\lambda_i} = g_{\lambda_i} = 1$ $\quad i=1,\dots,3$
- $k=-1$ $\quad a_0 = 2 \quad g_0 = 1$; $\quad a_2 = g_2 = 1$
- $k=1$ $\quad a_2 = 2 \quad g_2 = 1$; $\quad a_0 = g_0 = 1$

(b) A_k è diag. quando $k \neq \pm 1$.

(c) $k=0$
$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad P = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

ES.1: In $\mathbb{R}^3(\mathbb{R})$ det. una base $L\{v_1,v_2\}^\perp$

$v_1=(1,-1,1)$ e $v_2=(0,1,0)$, prod. scal. è così def.: $(x_1,y_1,z_1)\circ(x_2,y_2,z_2)=x_1x_2+y_1y_2+2z_1z_2$

• $L\{v_1,v_2\}^\perp=\{v_1,v_2\}^\perp$

$$\begin{cases}(x,y,z)\circ(1,-1,1)=0\\(x,y,z)\circ(0,1,0)=0\end{cases}\quad\begin{cases}x-y+2z=0\\0+y+0=0\end{cases}\quad\begin{cases}x=-2z\\y=0\end{cases}$$

$$\{v_1,v_2\}^\perp=\{(-2\alpha,0,\alpha)\mid\alpha\in\mathbb{R}\}=L((-2,0,1))$$

ES.2: In $\mathbb{R}^4(\mathbb{R})$ det. una base per $L(A)^\perp$

$A=\{(1,0,-1,0),(0,1,0,0)\}$ e il prod. scal. è così def.: $(x_1,y_1,z_1,t_1)\circ(x_2,y_2,z_2,t_2)=2x_1x_2+3y_1y_2+z_1z_2+2t_1t_2$

• $L(A)^\perp=A^\perp$

$$\begin{cases}(x\,y\,z\,t)\circ(1,0,-1,0)=0\\(x,y,z,t)\circ(0,1,0,0)=0\end{cases}\quad\begin{cases}2x+0-z+0=0\\3y=0\end{cases}\quad\begin{cases}z=2x\\y=0\end{cases}$$

$$A^\perp=\{(\alpha,0,2\alpha,\beta)\mid\alpha,\beta\in\mathbb{R}\}$$

$$B=((1,0,2,0),(0,0,0,1))$$

PRODOTTI SCALARI DEFINITI POSITIVI: $\forall v \neq \underline{0} \quad v \circ v > 0$

- NORMA: $\|v\| = \sqrt{v \circ v}$
- Due vett. $v, w \in \mathbb{R}^n$ hanno la STESSA DIREZ. se $v = \lambda \cdot w \quad \lambda \in \mathbb{R}$.
- VERSORE: $\|v\| = 1$.

● V: spz. vett. su $\mathbb{R}$

$\circ$: prod. scal. def. pos. in $V \quad w, v \in V(\mathbb{R}) \quad v \neq \underline{0}$

$$w = \boxed{\frac{v \circ w}{v \circ v}} v + \left(w - \frac{v \circ w}{v \circ v} v \right) = w_{/\!/} + w_{\perp}$$

COEFF. di FOURIER

ES. 3: In $\mathbb{R}^3(\mathbb{R})$ dati: $w = (3, -1, 3)$, $v = (0, 1, 1)$

il p.s. $(x_1, y_1, z_1) \circ (x_2, y_2, z_2) = x_1 x_2 + y_1 y_2 + 2 z_1 z_2$

Det. rispetto a B_c, la proiez. di w nella direz. di v ($w_{/\!/}$) e si scrive w come comb. lin. di $w_{/\!/}$ e di uno ortogonale a v.

COEFF. DI FOURIER: $\frac{v \circ w}{v \circ v} = 5/3$

$$v \circ w = (0,1,1) \circ (3,-1,3) = 3 \cdot 0 - 1 \cdot 1 + 2 \cdot 1 \cdot 3 = 5$$

$$v \circ v = (0,1,1) \circ (0,1,1) = 0 \cdot 0 + 1 \cdot 1 + 2 \cdot 1 \cdot 1 = 3$$

$$w_{\parallel} = \frac{v \circ w}{v \circ v} v = 5/3 \cdot (0,1,1) = (0, 5/3, 5/3)$$

$$w = (0, 5/3, 5/3) + \left((3,-1,3) - 5/3 \cdot (0,1,1)\right) =$$

$$= (0, 5/3, 5/3) + \underbrace{(3, -8/3, 4/3)}_{w_{\perp}}$$

$$\underset{w_{\perp}}{(3, -8/3, 4/3)} \circ \underset{v}{(0,1,1)} = -8/3 + 2 \cdot \frac{4}{3} = 0$$

ES4: In $\mathbb{R}^4(\mathbb{R})$ $u = (1,-1,1,0)$ $w = (1,0,1,1)$

p.s. $(x_1, y_1, z_1, t_1) \circ (x_2, y_2, z_2, t_2) = 2x_1x_2 + 3y_1y_2 + z_1z_2 + 2t_1t_2$

- $\|u\|$, $\|w\|$, la proiez. di u lungo w (u,w), si scrivano u e w come comb. lin. di VERSORI con la medesima direz., risp. u e w.

$\cdot\ \|u\| = \sqrt{(1,-1,1,0)\circ(1,-1,1,0)} = \sqrt{2\cdot 1\cdot 1 + 3(-1)\cdot(-1) + 1\cdot 1 + 2\cdot 0\cdot 0} =$

$= \sqrt{6}$

$\cdot\ \|w\| = \sqrt{(1,0,1,1)\circ(1,0,1,1)} = \sqrt{2+0+1+2} = \sqrt{5}$

$\cdot\ u_{\parallel}:\ \dfrac{w\circ u}{w\circ w}\, w = \dfrac{(1,0,1,1)\circ(1,-1,1,0)}{5}(1,0,1,1) =$

$= \dfrac{2+0+1+0}{5}(1,0,1,1) = \left(\frac{3}{5}, 0, \frac{3}{5}, \frac{3}{5}\right)$

$u = \sqrt{6}\cdot\left(\frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, 0\right)$

$w = \sqrt{5}\cdot\left(\frac{1}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$

- BASE ORTOGONALE: $B = (b_1, \dots, b_n)$ base di $\mathbb{R}^n$

  B è ORTOGONALE SSE $b_i \circ b_j = 0 \quad \forall i,j = 1,\dots n$
  risp. ad un p.s. $\qquad i \neq j$

- BASE ORTONORMALE; $B = (b_1 \dots b_n)$ ~~Base~~ di $\mathbb{R}^n$

  B è ORTONORM. SSE B è ORTOGON.
  risp. a p.S.D.P. $\qquad$ e $b_i \circ b_i = 1$
  $\forall i = 1 \dots n$

METODO DI ORTOGONALIZZAZIONE DI GRAM-SCHMIDT

Data una Base $B = (v_1 \_\_ v_n)$ di V.

$B_2 = (w_1 \_\_\_ w_n)$ è B. ORTOGONALE risp. a p.s.d.p.

- $w_1 = v_1$
- $w_2 = v_2 - \dfrac{v_2 \circ w_1}{w_1 \circ w_1} \cdot w_1$
- $w_3 = v_3 - \dfrac{v_3 \circ w_1}{w_1 \circ w_1} w_1 - \dfrac{v_3 \circ w_2}{w_2 \circ w_2} w_2$

$\vdots$

$$w_n = v_n - \sum_{i=1}^{n-1}$$

$$w_n = v_n - \sum_{i=1}^{n-1} \frac{v_n \circ w_i}{w_i \circ w_i} w_i$$

ES.5: In $\mathbb{R}^3(\mathbb{R})$ $B = (\underbrace{(1,0,0)}_{e_1}, \underbrace{(0,1,0)}_{e_2}, \underbrace{(0,0,1)}_{e_3})$

$(x_1, y_1, z_1) \circ (x_2\ y_2\ z_2) = x_1 x_2 + y_1 y_2 + z_1 z_2$

Base ORTONORMALE.

$\|e_1\| = \sqrt{(1,0,0) \circ (1,0,0)} = \sqrt{1} = 1$

$\|e_2\| = \ldots$

$$\|e_3\| = \sqrt{(0,0,1)\cdot(0,0,1)} = \sqrt{2\cdot 1\cdot 1} = \sqrt{2}$$

$$e_3 = \sqrt{2}\cdot\left(0,0,\frac{1}{\sqrt{2}}\right)$$

$$B_{ORTONORM.} = \left(e_1, e_2, \left(0,0,\frac{1}{\sqrt{2}}\right)\right)$$

$$B' = \left(\overbrace{(1,1,0)}^{e'_1}, \overbrace{(0,1,0)}^{e'_2}, \overbrace{(0,0,1)}^{e'_3}\right)$$

↓ ricavare una base ORTOGONALE e ORTONORMALE.

$$(1,1,0)\cdot(0,1,0) = 1 \neq 0$$

- $e'_1 = b_1$
- $b_2 = e'_2 - \dfrac{e'_2\cdot b_1}{b_1\cdot b_1}\, b_1 = (0,1,0) - \dfrac{1}{2}(1,1,0) =$

$$= \left(-\frac{1}{2}, \frac{1}{2}, 0\right)$$

$$b_3 = e_3' - \frac{e_3' \circ b_2}{b_2 \circ b_2} b_2 - \frac{e_3' \circ b_1}{b_1 \circ b_1} b_1 =$$

$$= (0,0,1) - \frac{0}{\cdots} - \frac{0}{\cdots} = (0,0,1)$$

$$\tilde{B} = \left((1,1,0), \left(-\tfrac{1}{2}, \tfrac{1}{2}, 0\right), (0,0,1)\right)$$

$\|b_1\| = \sqrt{b_1 \circ b_1} = \sqrt{2}$ $\qquad b_1 = \sqrt{2}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$

$\|b_2\| = \sqrt{b_2 \circ b_2} = \sqrt{\frac{1}{2}}$ $\qquad b_2 = \frac{1}{\sqrt{2}} \cdot \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right)$

$\|b_3\| = \sqrt{2}$ $\qquad b_3 = \sqrt{2} \cdot \left(0, 0, \frac{1}{\sqrt{2}}\right)$

<u>es</u>: $A = \begin{pmatrix} 3/2 & 0 & -1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & 3/2 \end{pmatrix}$

- λ: $\begin{pmatrix} 3/2-\lambda & 0 & -1/2 \\ 0 & 1-\lambda & 0 \\ -1/2 & 0 & 3/2-\lambda \end{pmatrix} \Rightarrow \det \ldots = (1-\lambda)^2 \cdot (2-\lambda)$

$\lambda_1 = 1 \quad a_1 = 2$

$\lambda_2 = 2 \quad a_2 = 1$

V_1: $(A - I_3) = \begin{pmatrix} 1/2 & 0 & -1/2 \\ 0 & 0 & 0 \\ -1/2 & 0 & 1/2 \end{pmatrix}$

$x = z$

$V_1 = \{(\alpha, y, \alpha) \in \mathbb{R}^3 \mid \alpha, y \in \mathbb{R}\}$

$$B_{V_1} = \big(\overbrace{(1,0,1)}^{e_1}, \overbrace{(0,1,0)}^{e_2}\big)$$

$$V_2: \ A - 2I_3 = \begin{pmatrix} -\frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & -1 & 0 \\ -\frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix} \quad \begin{cases} x = -z \\ y = 0 \end{cases}$$

$$B_{V_2} = \big(\underbrace{(-1,0,1)}_{e_3}\big)$$

- $$\underbrace{\begin{matrix}(1,0,1)\circ(-1,0,1) = 0 \\ (0,1,0)\circ(-1,0,1) = 0\end{matrix}}$$ questi autovettori (rel. ad autovalori distinti) sono ortog. (teor. spettrale).

- $$\underbrace{(1,0,1)\circ(0,1,0)} = 0$$ questi non è detto che lo siano; se non lo fossero useremmo il metodo di Gram-Schmidt per ortogonalizzarli-

$$\|e_1\| = \sqrt{e_1 \circ e_1} = \sqrt{(1,0,1)\circ(1,0,1)} = \sqrt{2}$$

$$\|e_2\| = \sqrt{e_2 \circ e_2} = 1$$

$$\|e_3\| = \sqrt{e_3 \circ e_3} = \sqrt{2}$$

$$e_1 = \sqrt{2} \cdot \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

$$e_3 = \sqrt{2} \cdot \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{2}\right)$$

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

P è ORTOGONALE

verifiche: $P^{-1} = P^t \Rightarrow P \cdot P^t = I_3$