Es6 . 3: $k = \pm 5$ $\lambda = -2$ $a_{-2} = 2$

$k = +5$ $\lambda = -2$

$\downarrow \dots$

$V_{-2} = \{(-3y - t, y, -t, t) \mid t, y \in \mathbb{R}\}$

$\rho_{-2} = 2 = a_{-2}$

A è diag.

$k = -5$ $\lambda = -2$

$\downarrow \dots$

$V_{-2} = \{(-3y, y, 0, 0) \mid y \in \mathbb{R}\}$

$\rho_{-2} = 1 = \dim V_2$

$\Downarrow$

A non è diag.

$k = 0$ $\lambda = 3$ $a_3 = 2$

$\downarrow \dots$

$V_3 = \{(x, y, 6x - 2y, 2y - x) \mid y, x \in \mathbb{R}\}$

$\rho_3 = 2$

A è diag.

ⓒ $K=-3$ $\quad A=\begin{pmatrix} -2 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & -3 & 3 \end{pmatrix}$

$\lambda_1 = 1$

$\lambda_2 = -2 \qquad a_{\lambda_i} = g_{\lambda_i} = 1 \quad (i=1\ldots 4) \Rightarrow A$ è diag.

$\lambda_3 = 6$

$\lambda_4 = 0 \qquad A' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad P=$

$V_1 = \{(0,y,0,0)\} = L((0,1,0,0))$

$V_{-2} = \{(-3y,y,0,0)\} = L((-3,1,0,0))$

$V_6 = \{(0,y,-5y,5y)\} = L((0,1,-5,5))$

$V_0 = \{(t,-2t,t,t)\} = L((1,-2,1,1))$

$$P = \begin{pmatrix} 0 & -3 & 0 & 1 \\ 1 & 1 & 1 & -2 \\ 0 & 0 & -5 & 1 \\ 0 & 0 & 5 & 1 \end{pmatrix}$$

ES.5 ; 05/11/09 - Prove interm.

$$A_k = \begin{pmatrix} k+1 & 0 & 3k \\ k & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \quad k \in \mathbb{R}.$$

ⓐ $\lambda(A_k)$ $a_{\lambda_i}$ e $g_{\lambda_i}$

ⓑ $k = ?$ $A_k$ è diag.

ⓒ $k=0$ $D$ simile $A_0$, $P$ diag.

ⓐ $\det(A_k - \lambda I_3) = \det\begin{pmatrix} k+1-\lambda & 0 & 3k \\ k & -1-\lambda & 2 \\ 0 & 0 & 1-\lambda \end{pmatrix} =$

$= (1-\lambda)(k+1-\lambda)(-1-\lambda)$

- $\lambda_1 = 1$
- $\lambda_2 = -1$
- $\lambda_3 = k+1$

- se $k \neq -2 \wedge k \neq 0$ $\lambda_i$ sono dist.

$a_1 = 1 = g_1$

$a_{-1} = 1 = g_{-1}$

$a_{k+1} = 1 = g_{k+1}$

• se $K=0$ $\quad \lambda_1=\lambda_3=1 \Rightarrow a_1=2$

$(\lambda_2=-1 \rightarrow a_{-1}=1=\rho_{-1})$

$V_1: (A_0-I_3)X=0$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & +2 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{spe.} \left\{ -2y+2z=0 \right. \quad y=z$$

$V_1=\{(x,y,y) \mid x,y\in\mathbb{R}\}$ $\dim V_1=2=\rho_1$

$L((1,0,0),(0,1,1))$

• se $K=-2$ $\quad \lambda_1=1 \quad a_1=\rho_1=1$

$\lambda_2=\lambda_3=-1 \quad a_{-1}=2$

$V_{-1}: (A_{-2}+I_3)X=0$

$$\begin{pmatrix} 0 & 0 & -6 \\ -2 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix} \Rightarrow \text{spe} \begin{cases} z=0 \\ -2x+2z=0 \end{cases} \begin{cases} z=0 \\ x=0 \end{cases}$$

$V_{-1}=\{(0,y,0) \mid y\in\mathbb{R}\}$ $\dim V_{-1}=1=\rho_{-1}$

ⓑ A è diag. se $K \neq -2$

ⓒ $(k=0)$ $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

$V_{+1} = L((1,0,0),(0,1,1))$

$V_{-1}: (A_0 + I_3)X = 0$

$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix} \to \begin{cases} x=0 \\ z=0 \end{cases}$ $V_{-1} = \{(0,y,0) \mid y \in \mathbb{R}\}$ $L((0,1,0))$

ES.1: $\mathbb{R}^3(\mathbb{R})$. Rispetto alla $B_c$, scrivere la matrice che rappresenta la seguente f.b.:

$(x_1, y_1, z_1) * (x_2, y_2, z_2) = 3x_1y_2 + y_1y_2 - 5z_1y_2 + y_1z_2$

$\cdot (0,0,1) * (0,1,0) \cdot = 3 \cdot 0 \cdot 1 + 0 \cdot 1 - 5 \cdot 1 \cdot 1 + 0 \cdot 0 = -5$

ⓐ scrivere la matr.: $A = \begin{array}{c} \\ x_1 \\ y_1 \\ z_1 \end{array} \begin{array}{c} \begin{array}{ccc} x_2 & y_2 & z_2 \end{array} \\ \begin{pmatrix} 0 & 3 & 0 \\ 0 & 1 & 1 \\ 0 & -5 & 0 \end{pmatrix} \end{array}$

ⓑ...

ES.2: $\mathbb{R}^2(\mathbb{R})$. Rispetto alla $B_c$ : $e_1(1,0)$ $e_2(0,1)$

$(1,0)*(1,0)=3 \quad (1,0)*(0,1)=-1$

$(0,1)*(1,0)=2 \quad (0,1)*(0,1)=0$.

$$A=\begin{array}{c|cc} & x_2 & y_2 \\ \hline x_1 & 3 & -1 \\ y_1 & 2 & 0 \end{array}. \qquad \bullet (1,2)*(-1,1)=?$$

$$(1,2)\begin{pmatrix}3 & -1\\ 2 & 0\end{pmatrix}\begin{pmatrix}-1\\ 1\end{pmatrix}=\ldots \quad ; (x_1,y_1)*(x_2,y_2)=$$

$$=3x_1x_2-1x_1y_2+2y_1x_2$$

$$\Rightarrow 3\cdot 1\cdot(-1)-1(1)\cdot(1)+2(2)\cdot(-1)=-8$$

- F.B. SIMM. = PRODOTTO SCALARE "$\circ$"

$$v\circ w=w\circ v \quad \forall v,w\in V$$

A è SIMMETRICA.

- $v$ è ORTOG. a $w$ : $v\circ w=0$.
- COMPL. ORTOGONALE : $A\subseteq V \quad A\neq\emptyset$

$$A^\perp=\{v\in V \mid v\circ a=0, \forall a\in A\}$$

PROPRIETÀ: ① $A^\perp$ è ssv di $V$

② $A^\perp = (L(A))^\perp$

③ $A \subseteq B \Rightarrow B^\perp \subseteq A^\perp$

④ $(A^\perp)^\perp = L(A)$

⑤ $V = A^\perp \oplus L(A)$

ES.3: $\mathbb{R}^2(\mathbb{R})$. rispetto $B_c$ f.b:

$$(x_1,y_1)\circ(x_2,y_2) = 4x_1x_2 - x_1y_2 - y_1x_2$$

ⓐ verificare che è simm.

ⓑ det. vettori ortogonali a se stessi

ⓒ $\{(3,2)\}^\perp$.

ⓐ $(x_1,y_1)\circ(x_2,y_2) = 4x_1x_2 - x_1y_2 - y_1x_2 =$

$$= 4\cdot x_2x_1 - y_2x_1 - x_2y_1 = (x_2,y_2)\circ(x_1,y_1)$$

$A = \begin{pmatrix} 4 & -1 \\ -1 & 0 \end{pmatrix}$ A è simm. $\Rightarrow$ "$\circ$" è simm.

b) $v \circ v = 0$

$(x,y)\circ(x,y) = 4xx - xy - xy = 2x(2x-y) = 0$

$\begin{cases} \underline{x=0} \\ \underline{y=2x} \end{cases}$

$v = (0,y)$

$x,y \in \mathbb{R}$

$v = (x, 2x)$

c) $\{(3,2)\}^{\perp} = \{v \in V \mid v \circ (3,2) = 0\}$

$(x,y)\circ(3,2) = 4\cdot x\cdot 3 - x\cdot 2 - y\cdot 3 =$

$\Rightarrow 10x - 3y = 0$

$y = \frac{10}{3}x$

$\{(3,2)\}^{\perp} = \{(x, \frac{10}{3}x) \mid x \in \mathbb{R}\}$

**ES.4** : $\mathbb{R}^2(\mathbb{R})$. risp. $B_c$ :

$$(x_1,y_1)\circ(x_2,y_2) = 2x_1x_2 - x_1y_2 - y_1x_2 + y_1y_2$$

$W^\perp = ?$ $\quad W = \{(x,2x) \mid x \in \mathbb{R}\}$

- $W^\perp = (L(W))^\perp \qquad (1,2) \Rightarrow W = L((1,2))$

$W^\perp$ : $(\alpha,\beta)\circ(1,2) = 0$

$$\cancel{2\cdot\alpha\cdot 1} - \cancel{\alpha\cdot 2} - \beta\cdot 1 + \beta\cdot 2 = 0$$

$$\beta = 0$$

$W^\perp = \{(\alpha,0) \mid \alpha \in \mathbb{R}\}$

- $(\alpha,\beta)\circ(x,2x) = 0$

$$\cancel{2\alpha x} - \cancel{\alpha\cdot 2x} - \beta x + \beta\cdot 2x = 0 \qquad (\forall x \in \mathbb{R})$$

$$\beta x = 0 \quad \Rightarrow \beta = 0$$

...

**ES.5:** $K \in \mathbb{R}$. $v = (k, 1, 2-k, 0) \in \mathbb{R}^4$

$$A = \{\underbrace{(1, -2, 1, 0)}_{a_1}, \underbrace{(0, 3, 1, 0)}_{a_2}\}$$

$K = ?$ $v \in A^{\perp}$. (pr. sc. def. comp. per comp.)
p.s.e.

- $v \in A^{\perp}$: $\begin{cases} v \circ a_1 = 0 \\ v \circ a_2 = 0 \end{cases}$ $\begin{cases} k\cdot 1 + 1\cdot(-2) + 1\cdot(2-k) + 0\cdot 0 = 0 \\ k\cdot 0 + 1\cdot 3 + (2-k)\cdot 1 + 0\cdot 0 = 0 \end{cases}$

$\begin{cases} 0 = 0 \\ k = 5 \end{cases}$ se $K = 5$ $v \in A^{\perp}$

**ES.6:** $\mathbb{R}^3(\mathbb{R})$. $v = (1, 0, 2)$. $\{v\}^{\perp}$: det. una B.

p.s.: $(x_1, y_1, z_1) \circ (x_2, y_2, z_2) = x_1 y_2 + x_2 y_1 - 2 z_1 z_2$
(risp. Bc)

- $\{v\}^{\perp}$: $(x, y, z) \in \mathbb{R}^3$ : $(x, y, z) \circ (1, 0, 2) = 0$

$\Rightarrow x \cdot 0 + 1 \cdot y - 2 z \cdot 2 = 0 \Rightarrow y = 4z$

$\{v\}^{\perp} = \{(x, 4z, z) \mid x, z \in \mathbb{R}\}$

$B_{v^{\perp}} = ((1, 0, 0) \text{ e } (0, 4, 1)) \Rightarrow \dim \{v\}^{\perp} = 2$

**ES.7:** $M_2(\mathbb{R})$:

"$\circ$": $\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \circ \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} = x_1 y_1 + 2 x_2 y_2 + x_4 y_4$

ⓐ $W^\perp$ ; $W = \left\{ \begin{pmatrix} x_1 & 0 \\ 0 & x_4 \end{pmatrix} \middle| \, x_1, x_4 \in \mathbb{R} \right\}$

ⓑ trovare vett. ortogonali a se stessi.

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ⓐ $W = L\left( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$

$$W^\perp \rightarrow \begin{cases} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \circ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 0 \\ \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \circ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0 \end{cases} \qquad \begin{cases} x_1 \cdot 1 + \cancel{2 \cdot x_2 \cdot 0} + \cancel{x_4 \cdot 0} = 0 \\ x_1 \cdot 0 + 2 x_2 \cdot 0 + x_4 \cdot 1 = 0 \end{cases}$$

$$\begin{cases} x_1 = 0 \\ x_4 = 0 \end{cases} \qquad W^\perp = \left\{ \begin{pmatrix} 0 & x_2 \\ x_3 & 0 \end{pmatrix} \middle| \, x_2, x_2 \in \mathbb{R} \right\}$$

$\downarrow$

$L\left( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right)$ $\dim W^\perp = 2$

ⓑ $\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \circ \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = x_1 x_1 + 2 x_2 x_2 + x_4 \cdot x_4$

$x_1^2 + 2x_2^2 + x_4^2 = 0 \iff x_1 = x_2 = x_4 = 0$

$\Rightarrow \begin{pmatrix} 0 & 0 \\ x_3 & 0 \end{pmatrix} \quad x_3 \in \mathbb{R}$.

<u>ES.8:</u> $V = \{(x,y,z) \mid x + y - 2z = 0\}$ s.s.v di $\mathbb{R}^3$

$B_V$ di vettori ORTOGONALI risp. al pr. scal. euclideo

- $x + y - 2z = 0 \qquad y = 2z - x$

$V = \{(x, 2z - x, z) \mid x, z \in \mathbb{R}\}$

. $v_1 = (1, -1, 0)$

troviamo $v_1^\perp$: $(x, 2z - x, z) \circ (1, -1, 0) = 0$

. $x - 2z + x = 0 \quad 2x - 2z = 0 \quad x = z$

$(x, y, x), x, y \in \mathbb{R} \quad v_2 = (1, 2, 1)$

$(x, y, x) \to v_2 = (1, 2, 1)$

$B_V$ ORTOG. $= (v_1, v_2)$

ES. 6 della p.i. 05/11/09

$\mathbb{R}^3(\mathbb{R})$ p.s.e. $A = \{ \overbrace{(1,0,-1)}^{a_1}, \overbrace{(-1,2,1)}^{a_2}, \overbrace{(0,1,0)}^{a_3} \}$

ⓐ vett. di A tra loro ortog.

ⓑ $A^\perp$.

ⓐ $a_1 \circ a_2 \neq 0$ $\quad a_1 \perp a_3$

$a_2 \circ a_3 \neq 0$

$a_1 \circ a_3 = 0$

$$\text{(b)} \begin{cases} (x,y,z)\circ(1,0,-1) = 0 \\ (x,y,z)\circ(-1,2,1) = 0 \\ (x,y,z)\circ(0,1,0) = 0 \end{cases} \quad \begin{cases} x - z = 0 \\ -x + \cancel{2y} + z = 0 \\ y = 0 \end{cases} \quad \begin{cases} 0 = 0 \\ x = z \\ y = 0 \end{cases}$$

$$A^\perp = \{(x,0,x) \mid x \in \mathbb{R}\} = L((1,0,1))$$