SPAZI E SOTTOSPAZI VETTORIALI

Dato un campo $\mathbb{K}$ e (V,+) gruppo abeliano, se è definita la legge di composizione tra gli scalari λ di $\mathbb{K}$ e gli elementi **v** di V tale che λ**v** appartenga a V e per ogni λ, $\mu \in \mathbb{K}$ e per ogni v, w $\in$ V valgono:

1) $(\lambda+\mu)\cdot \mathbf{v}=\lambda\cdot \mathbf{v}+ \mu\cdot \mathbf{v}$;
2) $\lambda\cdot(\mathbf{v}+\mathbf{w})= \lambda\cdot \mathbf{v}+\lambda\cdot \mathbf{w}$;
3) $(\lambda\, \mu)\cdot \mathbf{v}=\lambda\, (\mu\cdot \mathbf{v})$;
4) $1\cdot \mathbf{v}=\mathbf{v}$.

allora V è detto **spazio vettoriale** e i suoi elementi detti **vettori**.

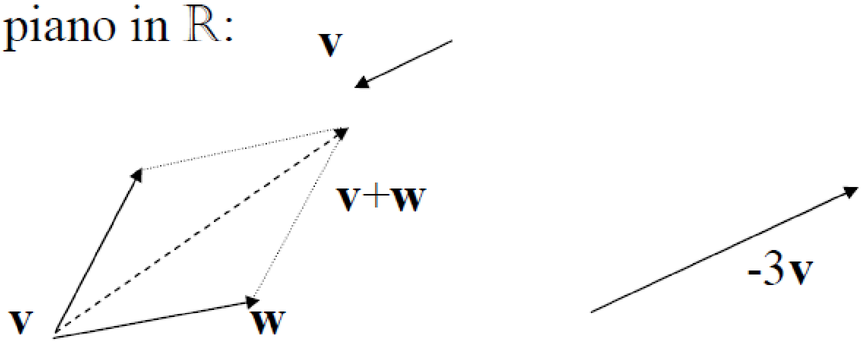
Esempi

1) Lo spazio vettoriale delle potenze di $\mathbb{R}$: $(\mathbb{R}^n,+,\cdot)$:

$$(x_1, \ldots, x_n)+(y_1, \ldots, y_n)=(x_1+y_1, \ldots, x_n+y_n)$$

$$\lambda(x_1, \ldots, x_n)=(\lambda x_1, \ldots, \lambda x_n)$$

2) lo spazio vettoriale geometrico $(V_2 , +,\cdot)$ sul piano in $\mathbb{R}$:



3) lo spazio vettoriale delle matrici $(\mathbb{R}^{m,n}, +, \cdot)$:

$$A+B=(a_{i,j})_{i\in I,\, j\in J} + (b_{i,j})_{i\in I,\, j\in J} = (a_{i,j}+b_{i,j})_{i\in I,\, j\in J}$$

$$\lambda A=\lambda(a_{i,j})_{i\in I,\, j\in J}=(\lambda a_{i,j})_{i\in I,\, j\in J}\,,\quad I=\{1,\dots,m\},\ J=\{1,\dots,n\}$$

Sottospazi di uno spazio vettoriale

1) Condizione necessaria e sufficiente affinché un insieme $U \neq \varnothing$, sottoinsieme di uno spazio vettoriale $(V(\mathbb{K}), +, \cdot)$, sia **sottospazio vettoriale** di $V(\mathbb{K})$:

a) $\forall\, u_1, u_2 \in U \quad \Rightarrow \quad u_1 + u_2 \in U$

b) $\forall\, \lambda \in \mathbb{K},\ \forall\, u \in U \quad \Rightarrow \quad \lambda u \in U$

2) Condizione necessaria e sufficiente affinché un insieme $U \neq \emptyset$, sottoinsieme di uno spazio vettoriale $(V(\mathbb{K}), +, \cdot)$, sia **sottospazio vettoriale** di $V(\mathbb{K})$:

$$\forall\, \alpha, \beta \in \mathbb{K},\ \forall \mathbf{u_1}, \mathbf{u_2} \in U \quad \Rightarrow \quad \alpha \mathbf{u_1} + \beta \mathbf{u_2} \in U$$

Esercizio 1

Nello spazio vettoriale $\mathbb{R}^3$ su $\mathbb{R}$ individuare quali dei seguenti insiemi sono sottospazi vettoriali:

$U_1 = \{(x,y,z) \in \mathbb{R}^3 \mid y + 5z = 0\}$;

$U_2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2= -1\}$;

$U_3 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2= 0\}$;

$U_4 = \{(x,y,z) \in \mathbb{R}^3 \mid x = -3\}$;

$U_5 = \{(1,2,1), (2,3,-1), (0,0,0)\}$.

(1) $U_1 = \{(x,y,z) \in \mathbb{R}^3 \mid y + 5z = 0\}$

$(0,0,0) \in U_1 \qquad U_1 \neq \emptyset$

$\forall \alpha, \beta \in \mathbb{R} \quad \forall u_1, u_2 \in U_1 \Rightarrow \alpha u_1 + \beta u_2 \in U_1$

$u_1 \Rightarrow \boxed{y_1 + 5z_1 = 0} \qquad u_2 : \boxed{y_2 + 5z_2 = 0}$

$u_1 \downarrow (x_1, y_1, z_1) \qquad u_2 \downarrow (x_2, y_2, z_2)$

$\alpha(x_1, y_1, z_1) + \beta(x_2, y_2, z_2) = (\alpha x_1, \alpha y_1, \alpha z_1) + (\beta x_2, \beta y_2, \beta z_2) =$

$= \underbrace{(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2)}$:

$\alpha y_1 + \beta y_2 + 5 \cdot (\alpha z_1 + \beta z_2) \overset{?}{=} 0$

$\alpha y_1 + \beta y_2 + 5\alpha z_1 + 5\beta z_2 = \alpha(\underbrace{y_1 + 5z_1}_{= 0}) + \beta(\underbrace{y_2 + 5z_2}_{= 0}) = 0$

U_1 è sottospazio vett. di $\mathbb{R}^3$.

(b) $U_2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = -1\}$

$(0,0,0) \notin U_2 \qquad U_2$ non è sottospazio vett.

ⓒ $U_3 = \{(x,y,z) \in \mathbb{R}^3 \mid \underbrace{x^2+y^2}_{\geq 0} = \underbrace{-z^2}_{\leq 0}\}$

$\downarrow$

$(0,0,0) \in U_3$ Sottospazio banale.

ⓓ $U_4 = \{(x,y,z) \in \mathbb{R}^3 \mid x = -3\}$

$(0,0,0) \notin U_4$ U_4 non è sottospazio vett.

ⓔ $U_5 = \{(1,2,1), (2,3,-1), (0,0,0)\}$

$(1,2,1)+(2,3,-1) = (3,5,0) \notin U_5$ non è sottospa vett.

Esercizio 2

Nello spazio vettoriale $M_2(\mathbb{R})$ verificare quali dei seguenti sottoinsiemi sono sottospazi vettoriali:

$$W_1 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\};$$

$$W_2 = \left\{ \begin{pmatrix} x & y \\ z & t \end{pmatrix} \mid x,y,z,t \in \mathrm{R} \wedge x = z = t = 0 \right\};$$

$$W_3 = \left\{ \begin{pmatrix} x & y \\ z & t \end{pmatrix} \mid x,y,z,t \in \mathrm{R} \wedge x = -1 \right\}$$

a) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \notin W_1$

W_1 non è sottospazio vett.

b) $W_2 = \left\{ \begin{pmatrix} x & y \\ z & t \end{pmatrix} \in M_2(\mathbb{R}) \;\middle|\; x,y,z,t \in \mathbb{R} \wedge x=z=t=0 \right\}$

$\begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \in W_2$ $\qquad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in W_2 \neq \emptyset$

$\forall \alpha, \beta \in \mathbb{R}, \forall \begin{pmatrix} 0 & y_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & y_2 \\ 0 & 0 \end{pmatrix} \in W_2$

$\alpha \begin{pmatrix} 0 & y_1 \\ 0 & 0 \end{pmatrix} + \beta \begin{pmatrix} 0 & y_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \alpha y_1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \beta y_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \alpha y_1 + \beta y_2 \\ 0 & 0 \end{pmatrix}$

$\in W_2$ è sottosp. vett.

$W_3 = \left\{ \begin{pmatrix} x & y \\ z & t \end{pmatrix} \in M_2(\mathbb{R}) \;\middle|\; x,y,z,t \in \mathbb{R} \wedge x = -1 \right\}$

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \notin W_3 \Rightarrow W_3$ non è sottospazio vett.

Copertura lineare di A in V($\mathbb{K}$)

Sia $A \subseteq V(\mathbb{K})$, $A \neq \varnothing$, si definisce copertura lineare L(A) il seguente insieme

$$L(A)=\left\{v \in V \;\middle|\; v=\alpha_1 v_1+\ldots+\alpha_n v_n \quad \forall\, \alpha_i \in K,\ v_i \in A\right\}$$

I vettori $\mathbf{v}_1,\ldots,\mathbf{v}_n$ si dicono **generatori** di L(A).

Il vettore v si dice **combinazione lineare** di $\mathbf{v}_1,\ldots\mathbf{v}_n$.

Esercizio 3

Verificare che l'elemento v appartenga alla copertura lineare di A in V:

a) $V=\mathbb{R}^4$ $A=\{(0,1,0,-3), (0,2,2,-1), (0,0,-1,0)\}$

$v=(0,3,1,-4)$;

b) $V=R^{2,3}$ $A=\left\{\begin{pmatrix}1 & 0 & 0\\ 1 & 0 & 0\end{pmatrix}, \begin{pmatrix}0 & -1 & 0\\ 2 & 0 & 0\end{pmatrix}\right\}$

$$v=\begin{pmatrix}3 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}$$

(a) $v=(0,3,+1,-4)\in L(A)$ sse
$\exists\ \alpha,\beta,\gamma\in\mathbb{R}$:

$$\alpha(0,1,0,-3)+\beta(0,2,2,-1)+\gamma(0,0,-1,0)=v$$

$$(0,\alpha,0,-3\alpha)+(0,2\beta,2\beta,-\beta)+(0,0,-\gamma,0)=$$

$$=(0,\alpha+2\beta,2\beta-\gamma,-3\alpha-\beta)=(0,3,1,-4)$$

$$\begin{cases}0=0\\ \alpha+2\beta=3\\ 2\beta-\gamma=1\\ +3\alpha+\beta=+4\end{cases}\quad\begin{cases}-\\ \alpha=3-2\beta\\ \gamma=2\beta-1\\ 9-6\beta+\beta=4\end{cases}\quad\begin{cases}\alpha=1\\ \gamma=1\\ \beta=1\end{cases}\qquad v\in L(A)$$

(b) $V=\mathbb{R}^{2,3}$ $\quad A=\left\{\begin{pmatrix}1&0&0\\1&0&0\end{pmatrix},\begin{pmatrix}0&-1&0\\2&0&0\end{pmatrix}\right\}$

$$v=\begin{pmatrix}3&1&0\\0&0&0\end{pmatrix}$$

$v\overset{?}{\in}L(A)$ sse $\exists\,\alpha,\beta\in\mathbb{R}$

$$\alpha\begin{pmatrix}1&0&0\\1&0&0\end{pmatrix}+\beta\begin{pmatrix}0&-1&0\\2&0&0\end{pmatrix}=\begin{pmatrix}\alpha&-\beta&0\\\alpha+2\beta&0&0\end{pmatrix}\overset{?}{=}\begin{pmatrix}3&1&0\\0&0&0\end{pmatrix}$$

$$\begin{cases}\alpha=3\\ \beta=-1\\ \alpha+2\beta\neq0\end{cases}\qquad \nexists\,\alpha,\beta\in\mathbb{R}\Rightarrow v\notin L(A)$$

Esercizio 4

Verificare che A sia un insieme di generatori di V nei seguenti casi:

a) $V=\{(\alpha,\alpha,\beta) \mid \alpha,\beta\in \mathbb{R}\}$

$A=\{(1,-1,0), (0,0,-1), (1,1,3)\}$;

b) $V=\left\{\begin{pmatrix} x & y \\ 0 & x \end{pmatrix} \mid x,y\in \mathrm{R}\right\} A=\left\{\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},\begin{pmatrix} 0 & -3 \\ 0 & 0 \end{pmatrix}\right\}$;

c) $V = R^{3,2}$

$$A=\left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix},\begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},\begin{pmatrix} 0 & -1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix},\begin{pmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 0 \end{pmatrix},\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 3 & 0 \end{pmatrix},\begin{pmatrix} 0 & 0 \\ 0 & 2 \\ 0 & 1 \end{pmatrix}\right\}$$

d) V del punto a) e $D=\{(1,1,0),(0,0,-1), (1,1,3)\}$.

(c) $L(A) = V \;\begin{cases} L(A) \subseteq V \\ V \subseteq L(A) \end{cases}$

• $\underbrace{V \subseteq L(A)}$ (α, α, β)

$x(1,-1,0) + y(0,0,-1) + z(1,1,3) =$

$= (x+z, -x+z, -y+3z) \overset{?}{=} (\alpha, \alpha, \beta)$

$\begin{cases} x+z=\alpha \\ -x+z=\alpha \\ -y+3z=\beta \end{cases} \quad \begin{cases} x=0 \\ z=\alpha \\ y=3\alpha-\beta \end{cases} \quad V \subseteq L(A)$

• $L(A) \not\subseteq V$ $(1,-1,0) \notin V$

A non è ins. di generatori per V.

(d) $L(D) \subseteq V$

$V \subseteq L(D)$: $(\alpha, \alpha, \beta) \in V$

$x(1,1,0) + y(0,0,-1) + z(1,1,3) = (x+z, x+z, -y+3z)$

$\begin{cases} x+z=\alpha \\ x+z=\alpha \\ -y+3z=\beta \end{cases} \quad \begin{cases} x=0 \\ z=\alpha \\ y=-\beta+3\alpha \end{cases} \Rightarrow V \subseteq L(D)$

$L(D) = V$

D è ins. di gen. per V.

b) $\left\{ \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} \in M_2(\mathbb{R}) \,\middle|\, x, y \in \mathbb{R} \right\} = V$

$A = \left\{ \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & -3 \\ 0 & 0 \end{pmatrix} \right\}$ $L(A) \subseteq V$

$V \subseteq L(A)$: $\exists\, \alpha, \beta \in \mathbb{R}$

$$\alpha \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} + \beta \begin{pmatrix} 0 & -3 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2\alpha & -3\beta \\ 0 & 2\alpha \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} x & y \\ 0 & x \end{pmatrix}$$

$\begin{cases} 2\alpha = x \\ -3\beta = y \end{cases}$ $\begin{cases} \alpha = \frac{x}{2} \\ \beta = -\frac{y}{3} \end{cases}$ $V \subseteq L(A) \Rightarrow L(A) = V$

A è ins. di gen. V

c) $V = \mathbb{R}^{3,2} = \left\{ \begin{pmatrix} x & y \\ z & t \\ u & v \end{pmatrix} \in \mathbb{R}^{3,2} \,\middle|\, x, y, z, t, u, v \in \mathbb{R} \right\}$

$$A = \left\{ \underset{\alpha}{\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}}, \underset{\beta}{\begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}}, \underset{\gamma}{\begin{pmatrix} 0 & -1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}}, \underset{\delta}{\begin{pmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 0 \end{pmatrix}}, \underset{\varepsilon}{\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 3 & 0 \end{pmatrix}}, \underset{\varphi}{\begin{pmatrix} 0 & 0 \\ 0 & 2 \\ 0 & 1 \end{pmatrix}} \right\}$$

- $L(A) \subseteq V$
- $V \subseteq L(A)$: $\exists\, \alpha, \beta, \gamma, \delta, \varepsilon, \varphi \in \mathbb{R}$

$$\alpha \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} + \beta \ldots = \begin{pmatrix} 2\beta & -\gamma \\ -\delta & 2\varphi \\ 3\varepsilon & \alpha+\gamma+\varphi \end{pmatrix} = \begin{pmatrix} x & y \\ z & t \\ u & v \end{pmatrix}$$

$$\begin{cases} \beta = \frac{x}{2} \\ \gamma = y \\ \delta = -z \\ \varphi = \frac{t}{2} \\ \varepsilon = \frac{u}{3} \\ \alpha + \gamma + \varphi = v \end{cases} \quad \rightarrow \quad \alpha = v + y - \frac{t}{2}$$

$V \subseteq L(A)$

$\Rightarrow$ A è ins. di gen. per V.

Esercizio 5

Trovare un insieme di generatori per i seguenti spazi vettoriali su $\mathbb{R}$:

a) $\mathbb{R}^4$;

b) $V=\{(\alpha,\alpha,0,\beta) \mid \alpha,\beta\in\mathbb{R}\}$;

c) $V=\{(\alpha,\beta,\gamma,\gamma) \mid \alpha,\beta,\gamma\in\mathbb{R}\}$;

d) $\mathbb{R}^{2,4}$.

a) $\mathbb{R}^4: \{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)\}$

d) $\mathbb{R}^{2,4}: \left\{ \begin{pmatrix} 1&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&1\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\1&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&1&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&1&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&1 \end{pmatrix} \right\}$

b) $V = \{(\alpha,\alpha,0,\beta) \mid \alpha,\beta \in \mathbb{R}\}$.

$(\alpha,\alpha,0,0)+(0,0,0,\beta) = \alpha(1,1,0,0)+\beta(0,0,0,1)$

$A = \{(1,1,0,0),(0,0,0,1)\}$

c) $V = \{(\alpha,\beta,\gamma,\gamma) \mid \alpha,\beta,\gamma \in \mathbb{R}\}$

$A = \{(1,0,0,0),(0,1,0,0),(0,0,1,1)\}$

Esercizio 6

I vettori (1,2,0), (1,-2,0),(1,0,1) sono generatori di $\mathbb{R}^3$?

$B=\{(1,2,0),(1,-2,0),(1,0,1)\}$ $L(B)=\mathbb{R}^3$

(x,y,z)

. $L(B)\subseteq\mathbb{R}^3$

. $\mathbb{R}^3\subseteq L(B)$ $\exists\alpha,\beta,\gamma\in\mathbb{R}$

$\alpha(1,2,0)+\beta(1,-2,0)+\gamma(1,0,1)=(\alpha+\beta+\gamma,\ 2\alpha-2\beta,\ \gamma)$

$$\begin{cases}\alpha+\beta+\gamma=x\\ 2\alpha-2\beta=y\\ \gamma=z\end{cases}\quad \begin{cases}\alpha+\beta+z=x & \alpha=x-z-\beta\\ 2x-2z-2\beta-2\beta=y\\ \gamma=z\end{cases}\quad \begin{cases}\alpha=\frac{2x-2z+y}{4}\\ \beta=\frac{y+2z-2x}{-4}\\ \gamma=z\end{cases}$$

$\mathbb{R}^3\in L(B)\Rightarrow B$ è ins. di gen.

Una sequenza di vettori $(\mathbf{v_1},\ldots,\mathbf{v_n})$ di $V(\mathbb{K})$ si dice **libera** e i vettori $\mathbf{v_1,\ldots,v_n}$ si dicono **linearmente indipendenti** se la loro generica combinazione lineare posta uguale al vettore nullo $\forall\alpha_1\ldots\alpha_n\in\mathbb{K}$

$\alpha_1\mathbf{v_1}+\ldots+\alpha_n\mathbf{v_n}=\underline{\mathbf{0}}$ implica $\alpha_1=\ldots=\alpha_n=0$

Esercizio 7

Quali delle seguenti sequenze sono libere in V?

a) $V=\mathbb{R}^5$ $A=((0,1,2,0,0), (0,1,-2,0,0), (0,1,0,1,0))$;

b) $V = R^{3,2}$

$$B = \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 2 \\ 0 & 1 \end{pmatrix} \right)$$

c) $V=\mathbb{R}^3$ $C=((1,1,0), (0,0,-3), (0,0,1))$;

a) $V=\mathbb{R}^5=\{(x,y,z,t,u)\in\mathbb{R}^5 \mid x,y,z,t,u\in\mathbb{R}\}$

$A=\{(0,1,2,0,0),(0,1,-2,0,0),(0,1,0,1,0)\}$

$\alpha,\beta,\gamma\in\mathbb{R}$

$\alpha(0,1,2,0,0)+\beta(0,1,-2,0,0)+\gamma(0,1,0,1,0)=(0,0,0,0,0)$

$(0,\alpha+\beta+\gamma,2\alpha-2\beta,\gamma,0)=(0,0,0,0,0)$

$$\begin{cases}\alpha+\beta=0\\ 2\alpha-2\beta=0\\ \gamma=0\end{cases}\quad \begin{cases}\boxed{\alpha=-\beta}\\ \beta=0\\ \gamma=0\end{cases}\quad \begin{cases}\alpha=0\\ \beta=0\\ \gamma=0\end{cases}\Rightarrow A \text{ è ins. LIBERO}$$

(b) $V = \mathbb{R}^{3,2}$

$$\mathcal{B} = \left\{ \underset{\alpha_1}{\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}}, \underset{\alpha_2}{\begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}}, \underset{\alpha_3}{\begin{pmatrix} 0 & -1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}}, \underset{\alpha_4}{\begin{pmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 0 \end{pmatrix}}, \underset{\alpha_5}{\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 3 & 0 \end{pmatrix}}, \underset{\alpha_6}{\begin{pmatrix} 0 & 0 \\ 0 & 2 \\ 0 & 1 \end{pmatrix}} \right\}$$

$$\alpha_1 \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} + \alpha_2 \left(\quad \cdots\cdots \quad \right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 2\alpha_2 & -\alpha_3 \\ -\alpha_4 & 2\alpha_6 \\ 3\alpha_5 & \alpha_1+\alpha_3+\alpha_6 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \begin{cases} \alpha_2=\alpha_3=\alpha_4=\alpha_6=\alpha_5=0 \\ \alpha_1 = 0 \end{cases}$$

$\mathcal{B}$ è libero

(c) $V = \mathbb{R}^3$ $\quad C = \{(1,1,0),(0,0,-3),(0,0,1)\}$

$$\alpha\cdot(1,1,0) + \beta(0,0,-3) + \gamma(0,0,1) = (0,0,0)$$

$$(\alpha, \alpha, -3\beta+\gamma) = (0,0,0)$$

$$\begin{cases} \alpha = 0 \\ -3\beta+\gamma = 0 \end{cases} \qquad \begin{cases} \\ \boxed{\gamma = 3\beta} \end{cases}$$

C è legato

$$(0,0,-3) = -3\cdot(0,0,1)$$

Esercizi da svolgere

1. Quali tra i seguenti insiemi sono sottospazi vettoriali di V (motivare nel caso la risposta con opportuna dimostrazione):

   a) $V=\mathbb{R}^4$ $W=\{(x_1,x_2,x_3,x_4)\in V \mid x_1-3x_3=0,\ x_4-x_2=0\}$

   b) $V=M_3(\mathbb{R})$ $W=\{(a_{i,j})\in V \mid i<j,\ a_{i,j}=0\}$.

2. Mostrare che la copertura in V dell'insieme A è W:

   a) $V=\mathbb{R}^4$ $A=\{(1,0,-1,0),(0,2,3,0),(1,1,0,0)\}$

   $W=\{(x_1,x_2,x_3,x_4)\in V \mid x_4=0\}$

   b) $V=\mathbb{R}^4$ $A=\{(1,0,-1,0),(3,0,-3,0),(0,1,0,0)\}$

   $W=\{(x_1,x_2,x_3,x_4)\in V \mid x_1+x_3=0,\ x_4=0\}$

   c) $V=M_2(R)$ $W=M_2(R)$

   $$A=\left\{\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}$$

3. Nell'esercizio 2 trovare altri due insiemi di generatori oltre ad A per il s.s.v. W.

4. Quali tra gli insiemi A dell'esercizio 2 sono liberi?